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Effect of Collisions on Cold Ion Collection by Means of Langmuir Probes

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Langmuir probes are extensively used to sample plasma densities. However, the interpretation of probe data is well established only for the limiting cases of collisionless and collision-dominated plasmas. A transitional regime prevails in some fluid-dynamic situations and electrical discharges in which the ions are colder than the electrons. A simple theory which accounts separately for the effects of collisions between charged particles and collisions with neutrals is presented. When the theory is applied to recent experimental data taken with cylindrical as well as spherical probes, the electron densities calculated from the probe measurements by using the present theory agree within 10% with those inferred from existing numerical solutions, an empirical formula, and microwave cavity data.

Nomenclature

a = radius at last collision
c = aspect ratio
e = electronic charge
E = electric field
I = electric current
I_D = discharge current
J = normalized ion current
k = Boltzmann constant
L = focal distance
L_p = probe length
m = particle mass
n = normalized number density
N = number density
r = spherical radial coordinate
T = temperature
u = radial velocity component
v = normalized ion velocity
V = velocity
w = azimuthal velocity component
x = normalized inverse radius
Z = number of electronic charges
z = cylindrical axial coordinate
α = spherical correction index
β = cylindrical correction index
γ = normalized collision frequency
ζ = spheroidal length coordinate
η = normalized potential
λ = mean free path
λ_D = Debye length
μ = spheroidal angular coordinate
ν = collision frequency
ξ = normalized inverse radius
ρ = cylindrical radial coordinate
φ = potential
ω_p = plasma frequency

Superscripts

o = collisionless A.B.R. theory

Subscripts

c = cylindrical
ca = cavity measurement
e = electron
i = ion
n = neutral
o = sheath edge
p = probe
s = spherical
∞ = far-away from the probe

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I. Introduction

Langmuir probes are extensively used to sample plasma densities. For this purpose ion collection is preferable over electron collection because in the former case, the current drawn and, therefore, the perturbation introduced by the probe are much smaller. For a given potential difference between the probe and the plasma, the collected ion current depends strongly on the temperature of the electrons as well as the ions. In most electrical discharges and in many hypersonic testing facilities the ions are much colder than the electrons. As a consequence a great deal of attention has been paid to this particular situation.

Relatively good agreement between theory and experiment has been obtained with spherical probes especially in the limiting cases of collisionless and collision-dominated plasmas. Such good agreement has not been obtained with the more widely used cylindrical probes. With this geometry and in the presence of ion-neutral collisions a theoretical analysis encounters difficulties, because the assumption of cylindrical symmetry leads to a problem which is not well posed mathematically. In the absence of collisions there exist two mathematically consistent theories. 1) The macroscopic theory of Allen, Boyd, and Reynolds[1] [(A.B.R., see also Ref. 2)] predicts that the ion saturation current density increases monotonically with decreasing probe radius. 2) The microscopic theory of Bernstein and Rabinowitz[3] [(B.R., see also Refs. 4 and 5)] predicts that the ion saturation current should saturate for probe radii smaller than the sheath thickness, as a result of the orbital motion of the particles. Experimental data taken by Sonin[6] do not exhibit the expected saturation effect. Nor has it been observed in later experiments conducted under quite different plasma conditions by Self and Shih[7] and Lederman, Bloom and Widhopf.[8]

The present investigation is aimed at detecting the effect of collisions on ion collection, by studying the transitional regime where the number of collisions occurring within the space-charge-sheath enshrouding the probe is small but finite.

We begin by considering the spherical probe, a case in which collisionless theories are well established and give substantially the same results in the limit of small ion temperature, since orbital motion is not important. The effect of collisions on the determination of plasma density was already considered by one of these authors (Shih).[7] However, the procedure involved was based on numerical computations. We seek here, instead, a simpler, approximate analytical method where the collision frequency is treated as a small parameter in a perturbation scheme. In essence we neglect the collision effects within the space-charge-sheath but we take into account the ohmic drop in the quasi-neutral plasma region.

Friedman and Levi[9] have shown that the velocity with which the ions enter the space-charge-sheath does not depend on the amount of collisions which occur in the quasi-neutral plasma region. If in addition the space-charge-sheath is considered collisionless, the radius of the plasma-sheath interface, and therefore the plasma density at that point, are also independent of collisions and can be determined with the help of a collisionless theory, for instance the A.B.R. theory.[1,2] If we now assume that in the quasi-neutral plasma region the common electron and ion densities are related to the electric potential by the Boltzmann factor, the desired plasma density far-away from the probe can be obtained simply by determining the voltage drop across the quasi-neutral plasma region. This is the collisionless drop, which according to Lam[10] corresponds to one half the electron thermal energy, augmented by the ohmic drop resulting from collisions. The latter is adequately approximated by using the velocity profile corresponding to the collisionless theory.

The cylindrical probe is considered next. In this case, due to the two-dimensional character of the problem, the electric field extends to larger radii than with spherical probes and the likelihood of orbital motion is greater. The orbital motion can be taken correctly into account only by a microscopic theory, such as the B.R. theory. However, since the Coulomb cross section for ion-ion collisions increases rapidly as the temperature decreases, in the limit of small ion temperature, there always exist ion-ion collisions in sufficient numbers to destroy the orbital motion. In the presence of collisions the angular momentum of the individual particles is not conserved; the ions, while approaching the probe, fail to acquire sufficient transverse velocity to perform orbital motion and are therefore collected. As a result the B.R. theory underestimates the ion saturation current, even in the absence of collisions with neutrals. The A.B.R. theory, on the other hand, being macroscopic, takes correctly into account ion-ion collisions which do not affect the overall conservation of momentum of the species. However, in the presence of collisions with neutrals, it overestimates the ion saturation current. For this reason we base our calculations on the A.B.R. theory and introduce the effect of collisions with neutrals, by using the same perturbation techniques as for the spherical probe. Moreover, in order to make the problem mathematically well posed, we approximate the finite-length cylindrical probe by an ellipsoid of revolution, i.e., a spheroid.

II. Theory for Spherical Probes

In this section we develop a theory for spherical probes. Our treatment refers to a plasma satisfying the following conditions: 1) the plasma is infinite in extent, 2) the probe is operated in the ion saturation regime, so that the electron density has a Boltzmann profile, 3) the mean free path, either for ion-ion, or ion-neutral encounters is smaller than the sheath thickness and satisfies an inequality derived in Appendix A, and 4) collisions with neutrals do not exceed an upper bound which is defined in Appendix B.

Under these circumstances the ion dynamics can be described[7] by a set of three macroscopic equations: a) a continuity equation in which the generation and recombination terms are ignored, b) a momentum equation in which the pressure term $\nabla \cdot \mathcal{P}$ is dropped and the collision integral is set equal to a constant collision frequency ν_{in} times the average momentum density, and (c) Poisson's equation. This set is then given[7] by

$$\nabla \cdot N_i \mathbf{V}_i = 0$$
$$m_i N_i (\mathbf{V}_i \cdot \nabla)\mathbf{V}_i = -eZN_i \nabla\varphi - \nu_{in} m_i N_i \mathbf{V}_i \tag{1}$$
$$\nabla^2 \varphi = -4\pi e(ZN_i - N_e)$$

We introduce the following standard normalization and coordinate transformation

$$\eta = e\varphi/kT_e; \qquad v = -V_i/(ZkT_e/m_i)^{1/2}; \qquad n = ZN_i/N_{e\infty}{}^{o}$$
$$\xi = r/\lambda_D; \qquad x = (2^{1/2}J)^{1/2}/\xi; \qquad \gamma = \nu_{in}/\omega_{pi}$$

where

$$\lambda_D = (kT_e/4\pi N_{e\infty}{}^{o} e^2)^{1/2}$$
$$\omega_{pi} = (4\pi N_{e\infty}{}^{o} e^2 Z/m_i)^{1/2}$$
$$J = (m_i/2Z)^{1/2} eI_i/(kT_e)^{3/2}$$

By introducing a spherical coordinate system and by letting $N_e = N_{e\infty}{}^{o} \exp(e\varphi/kT_e)$ and $N_i = -I_i/4\pi r^2 eZV_i$, we obtain

$$dv/dx = [-x^2 d\eta/dx - \gamma(2^{1/2}J)^{1/2}v]/x^2 v \tag{2}$$
$$d^2\eta/dx^2 = -(2^{1/2}J)(x^2 - v\exp\eta)/x^4 v \tag{3}$$

These two equations have been solved numerically in Ref. 7 for various values of J and γ subject to the boundary conditions at infinity ($x = 0$)

$$v = \eta = \eta' = 0$$

If we introduce the series expansion

$$v = \sum_{n=0}^{\infty} a_n x^n$$
$$\eta = -\sum_{n=0}^{\infty} b_n x^n$$

we find that the first few coefficients of the series obtained from Eqs. (2) and (3) are

$$a_o = a_1 = 0 \qquad a_2 = 1 \qquad a_3 = (2^{1/2}J)^{1/2}\gamma$$

$$a_4 = (2^{1/2}J)\gamma^2 \qquad a_5 = (2^{1/2}J)^{3/2}\gamma^3 \qquad a_6 = (2^{1/2}J)^2\gamma^4 + \tfrac{1}{2} - \gamma^2$$

$$b_o = 0 \qquad b_1 = (2^{1/2}J)^{1/2}\gamma \qquad b_2 = \tfrac{1}{2}(2^{1/2}J)\gamma^2$$

$$b_3 = \tfrac{1}{3}(2^{1/2}J)^{3/2}\gamma^3 \qquad b_4 = \tfrac{1}{4}(2^{1/2}J)^2\gamma^4 + \tfrac{1}{2}$$

In the limit of zero ion-neutral collisions the system of equations reduces precisely to that of the A.B.R. theory and the coefficients become

$$a_o^o = a_1^o = 0 \qquad a_2^o = 1 \qquad a_3^o = 0$$

$$b_o^o = b_1^o = b_2^o = b_3^o = 0 \qquad b_4^o = \tfrac{1}{2}$$

We observe that in this case the first significant term in η is b_4^o, so that the potential decays asymptotically as r^{-4}. In the collisional case, instead, the first significant term is b_1, thus indicating a much slower decay, proportional to r^{-1}. With regard to velocity, however, collisions seem to have a much smaller effect because in both cases the velocity decays asymptotically as r^{-2}. In seeking solutions which are valid in the presence of a small amount of collisions, we are, thus, prompted by these observations to construct a perturbation scheme in which the collisionless v^o appears as the zeroth order solution and the collisional term $\gamma(2^{1/2}J)^{1/2}v$ in Eq. (2) is treated as a small perturbation.

If we define as the sheath edge the radius at which the space charge becomes appreciable, we know from the work of Schulz and Brown[11] that its position is not affected by a slight amount of collisions not exceeding the limit set by Eq. (B2) of Appendix B. Hence, denoting by a subscript o the value which the variables attain at the sheath edge, we have $r_o \doteq r_o^o$ or $x_o \doteq x_o^o$. We also know from the work of Friedman and Levi[9] that, irrespective of collisions, the directed velocity of the ions at the sheath-edge must satisfy Bohm criterion, i.e.,

$$V_{io} = V_{io}^o = -[k(ZT_e + T_i)/m_i]^{1/2}$$

or for cold ions

$$V_{io} \doteq -(ZkT_e/m_i)^{1/2} \qquad v_o^o = 1$$

Integrating Eq. (2) with respect to x from zero ($r \to \infty$) to the sheath edge $x_o(r = r_o)$ and introducing the collisionless values for x_o and v_o we obtain

$$\eta_o = -(\tfrac{1}{2})v_o^{o2} - \alpha = -\tfrac{1}{2} - \alpha \tag{4}$$

where

$$\alpha = \gamma(2^{1/2}J)^{1/2}\int_0^{x_o^o}(v/x^2)dx$$

Since we have assumed that collisions cause only a minor perturbation, we can evaluate α by substituting v^0 for v. Integration of the series solution term by term then yields

$$\alpha = \gamma(2^{1/2}J)^{1/2}x_o^o\{1 + \tfrac{1}{10}x_o^{o4} + \tfrac{1}{7}(\tfrac{1}{2} + \tfrac{6}{2}^{1/2}J)x_o^{o6} + \ldots\} \doteq \gamma(2^{1/2}J)^{1/2}x_o^o \tag{5}$$

The position of the sheath edge x_o^o can be obtained from the continuity relation with the assumption of neutrality in the region outside the sheath

$$I_i = -4\pi r_o^{o2}eZN_{io}V_{io}^o = -4\pi r_o^{o2}eV_{io}^oN_{e\infty}^o\exp\eta_o^o \tag{6}$$

and

$$x_o^o = (v_o^o\exp\eta_o^o)^{1/2} = v_o^{o1/2}\exp(-v_o^{o2}/4) = 0.78$$

It appears, then, that the position of the sheath edge x_o^o is rather insensitive to the assumed value of the ion velocity v_o^o and that the neglected higher order terms contribute less than 5% to the value of α.

In physical terms the correction parameter α represents a normalized ohmic drop due to ion-neutral collisions. Reintroducing dimensional quantities this ohmic drop can be written as

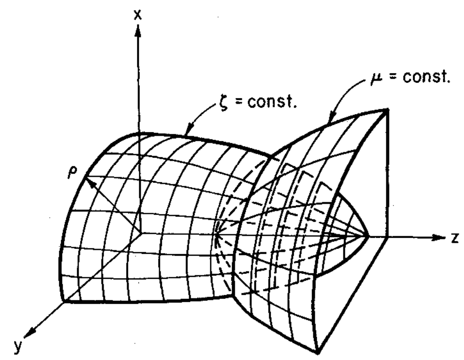

Fig. 1 Spheroidal coordinates.

$$\alpha kT_e/e = (\nu_{in}m_i/e^2N_{e\infty}^oZ)(r_o^o/4\pi r_o^{o2})I_i$$

where $\nu_{in}m_i/e^2ZN_{e\infty}^o$ thus assumes the role of an equivalent resistivity.

Once α has been evaluated it is possible to correct the value of plasma density $N_{e\infty}^o$ obtained from the collisionless A.B.R. theory by making use again of the continuity relation

$$I_i = -4\pi r_o^{o2}eV_{io}N_{e\infty}^\alpha\exp\eta_o \tag{7}$$

A comparison of Eqs. (6) and (7) with $V_{io} = V_{io}^o$, then, yields according to Eq. (4)

$$N_{e\infty}^\alpha = N_{e\infty}^o\exp(\eta_o^o - \eta_o) = N_{e\infty}^o\exp\alpha \tag{8}$$

III. Theory for Cylindrical Probes

We now turn our attention to the more commonly used cylindrical probes. Again we assume that the plasma is described by Eqs. (1) but, in order to take properly into account the effect of finite length of the probe we use a spheroidal coordinate system.[12] A probe of twice the actual length of the cylindrical probe and collecting twice as much current is approximated by a prolate spheroid and fitted into a constant coordinate surface which belongs to a family of confocal prolate spheroids (Fig. 1). This is done by choosing an appropriate aspect ratio. Furthermore we assume that the potential distribution varies along the length coordinate ζ only and is independent of the angular coordinate μ. Similar assumptions have been made by Cohen[13] and Su and Kiel[14] for collision-dominated plasma. However, whereas they assume that the density and ion velocity are likewise independent of μ, we will find from Poisson equation that this last assumption is unsustainable and, therefore, we retain the μ-dependence in N_i and V_i. To justify our assumption that the potential depends on ζ only, we observe that this is strictly the case at both extremes: far-away from the probe, since there the equipotential surfaces approach a spherical form, and in the neighborhood of the probe, since the probe surface, which is assumed to coincide with one of the prolate spheroids, is an equipotential surface. With regard to the ion velocity, we assume that $\mathbf{V}_i$ is parallel to the electric field $\mathbf{E}$. The assumed motion of the ions along the lines of force of $\mathbf{E}$ implies that we rely on collisions to impede orbital motion and other inertia related effects. Significant departures from the assumed trajectories are likely to occur only in those regions near the probe where the curvature of its surface undergoes a rapid change. Our analysis will show that, as with spherical probes, the collision correction factor, which we are seeking, is proportional to the total energy lost by the ions to the neutrals along their entire trajectory up to the sheath, i.e.,

$$\int_\infty^{\zeta_0} m_i\nu_{in}V_i d\zeta$$

In the case of a cylindrical probe the electric field penetrates deep into the plasma and, therefore, the main contribution to this integral comes from the region far from the probe. The assumptions made with regard to ion velocity, therefore, do not introduce significant error.

In order to express the fundamental set of equations, Eqs. (1), in spheroidal coordinates we introduce first a cylindrical system with radius ρ and axial coordinate z. In this system the equations describing confocal prolate spheroids are

$$\rho^2/(\zeta^2 - 1) + z^2/\zeta^2 = L^2/4 \tag{9}$$

where $\zeta(\geq 1)$ is the spheroidal coordinate of length and L is the focal distance corresponding in our case to about twice the probe length $[L \doteq 2L_p(1 - 1/c_p^2)]$ with c_p being the aspect ratio of the probe. The confocal orthogonal surfaces, hyperboloids of two sheets are given by

$$-\rho^2/(1 - \mu^2) + z^2/\mu^2 = L^2/4 \tag{10}$$

where μ is the angular spheroidal coordinate ($|\mu| \leq 1$) and the metrical coefficients are

$$h_1 = (L/2)[(\zeta^2 - \mu^2)/(\zeta^2 - 1)]^{1/2}$$
$$h_2 = (L/2)[(\zeta^2 - \mu^2)/(1 - \mu^2)]^{1/2}$$
$$h_3 = (L/2)(\zeta^2 - 1)^{1/2}(1 - \mu^2)^{1/2}$$

The aspect ratio c_ζ of the ζ_c spheroid is defined as

$$c_\zeta = z_{\max}/\rho_{\max} = \zeta_c/(\zeta_c^2 - 1)^{1/2} \tag{11}$$

We introduce the same normalization scheme as for the spherical probe. This normalization and the assumptions made with regard to the potential and the ion velocity lead to the following set of equations governing the dynamics of the ions

$$(\partial/\partial)\zeta[(\zeta^2 - \mu^2)^{1/2}(\zeta^2 - 1)^{1/2}nv] = 0 \tag{12}$$

$$(2\lambda_D/L)v(\partial v/\partial\zeta) = -(2\lambda_D/L)(\partial\eta/\partial\zeta) + \gamma[(\zeta^2 - \mu^2)/(\zeta^2 - 1)]^{1/2}v \tag{13}$$

$$[1/(\zeta^2 - \mu^2)](\partial/\partial\zeta)[(\zeta^2 - 1)(\partial\eta/\partial\zeta)] = (L^2/4\lambda_D^2)\ (\exp\eta - n) \tag{14}$$

Since the left hand side of Poisson's equation (Eq. 14) contains μ and, according to our assumptions, η is a function of ζ only, we conclude than n depends on μ.

Equation (12) can be integrated over ζ to give

$$G(\mu) = (\zeta^2 - \mu^2)^{1/2}(\zeta^2 - 1)^{1/2}n(\zeta,\mu)v(\zeta,\mu) \tag{15}$$

$G(\mu)$ is, thus, an unknown function independent of ζ which can be related to the total ion current I_i by integrating again over μ. Since generation and annihilation of ions have been neglected the total current entering any closed surface enclosing the probe is conserved. In particular we choose the spherical surface at infinity where $r = L\zeta/2$, and obtain

$$\int_{-1}^{1} G(\mu)d\mu = \int_{-1}^{1} (\zeta^2 - \mu^2)^{1/2}(\zeta^2 - 1)^{1/2}n(\zeta,\mu)v(\zeta,\mu)d\mu = \lim_{\zeta = 2r/L\to\infty} \int_{-1}^{1} \zeta^2 n(\zeta)v(\zeta)d\mu \tag{16}$$

We observe that on the infinite sphere n and v depend only on ζ and therefore the integration is straightforward and, with $2r/L$ replacing ζ, yields

$$\int_{-1}^{1} G(\mu)d\mu = \lim_{\zeta = 2r/L\to\infty} 2\zeta^2 n(\zeta)v(\zeta) = \frac{I_i}{\frac{1}{2}\pi L^2 N_{e\infty}{}^o e(ZkT_e/m_i)^{1/2}} \tag{17}$$

Equation (12) in the set of governing equations can now be replaced by

$$\int_{-1}^{1} (\zeta^2 - \mu^2)^{1/2}(\zeta^2 - 1)^{1/2}n(\zeta,\mu)v(\zeta,\mu)d\mu = \frac{I_i}{\frac{1}{2}\pi L^2 N_{e\infty}{}^o e(ZkT_e/m_i)^{1/2}} \tag{18}$$

In order to determine the plasma density we proceed, as in the case of the spherical probe, with a perturbation scheme based on the collisionless solution. By letting $\gamma = 0$, and assuming $v = \eta = \eta' = 0$ at infinity, we obtain from Eq. (13):

$$v^o = (-2\eta^o)^{1/2} \tag{19}$$

Introducing this relation and $n = e^{\eta^o}$ Eq. (18) can be rewritten for the quasi-neutral region as

$$\int_{-1}^{1} (\zeta^2 - \mu^2)^{1/2}(\zeta^2 - 1)^{1/2}e^{\eta^o}(-2\eta^o)^{1/2}d\mu = \frac{I_i}{\frac{1}{2}\pi L^2 e N_{e\infty}{}^o(ZkT_e/m_i)^{1/2}}$$

and, finally, integrating and normalizing as

$$2^{1/2}J = \tfrac{1}{8}e^{\eta^o}(-2\eta^o)^{1/2}(L/\lambda_D)^2[(\zeta^2 - 1)^{1/2} + \zeta^2 \sin^{-1}(1/\zeta)] \tag{20}$$

We then assume, as for the case of the spherical probe, a series solution and replace η^o in Eq. (20) by

$$\eta^o = -\sum_{n=0}^{\infty} b_n{}^o(1/\zeta)^n$$

The collisionless potential in the plasma region $\zeta > 1$ is then:

$$\eta^o \doteq -(2^4\lambda_D{}^4J^2/L^4)[1/\zeta^4 + 4/3\zeta^6 + \ldots.] \tag{21}$$

According to Eq. (19) the velocity is:

$$v^o = (2^{5/2}\lambda_D{}^2J/L^2)[1/\zeta^2 + 2/3\zeta^4 + \ldots.] \tag{22}$$

Replacing ζ by $2r/L$ we obtain the asymptotic solution for the potential

$$\eta^o \sim -J^2(r^4/\lambda_D{}^4) = -x^4/2 \tag{23}$$

This, of course, agrees, as it should, with the potential of the spherical probe, since for $r \gg L$ both the ellipsoidal and spherical probes look like a point probe.

The collision correction factor is obtained next, by following the same procedure as for the spherical probe. Integrating Eq. (13) we obtain, similarly to Eq. (4),

$$\eta_o = -\tfrac{1}{2}v_o{}^{o2} - \beta$$

where

$$\beta = \frac{\gamma L}{2\lambda_D}\int_{\infty}^{\zeta_o{}^o} \left(\frac{\zeta^2 - \mu^2}{\zeta^2 - 1}\right)^{1/2} v d\zeta \tag{24}$$

The upper limit $\zeta_o{}^o$ corresponds to the value of ζ at the sheath edge. The corresponding $\rho_o{}^o$ can be estimated by approximating the sheath surface to be cylindrical and by making use of the continuity equation

$$2\pi\rho_o{}^o e V_{io}{}^o L N_{e\infty}{}^o \exp\eta_o{}^o = -I_i$$

If $\eta_o{}^o$ is known, one can determine $\rho_o{}^o$ and hence $\zeta_o{}^o$ making use of Eq. (11)

$$\zeta_o{}^o = [(2\rho_o{}^o/L)^2 + 1]^{1/2}$$

Since $2\rho_o{}^o/L$ is likely to be in the order of 0.1 or less, $\zeta_o{}^o \doteq 1$. Expanding $(\zeta^2 - \mu^2)^{1/2}/(\zeta^2 - 1)^{1/2}$ as $1 - (\mu^2 - 1)/2\zeta^2 \ldots$ and integrating Eq. (24) term by term, with v replaced by v^o, we finally obtain

$$\beta = (2^{3/2}\gamma\lambda_D J/L)(25/18 - \mu^2/6 + \ldots) = (\tfrac{25}{9}2^{1/2})\gamma J\lambda_D/L \tag{25}$$

Table 1 Cylindrical probe measurements in an electrical discharge[a]

p mm	ρ_p 10^{-2}cm	T_e 10^3°K	ν_{in} 10^6/sec	$I(\eta_p = -10)$ ma	ρ_p/λ_D	$N_{e\infty}{}^0$ 10^{10}/cc	$-V_{io}$ 10^6cm/sec	e^β	$N_{e\infty}{}^\beta$ 10^{10}/cc
	3.81	...	...	3.00	2.00	1.81	...	1.15	2.09
0.05	1.27	136	1.3	2.05	0.76	1.96	1.66	1.07	2.10
	0.635	...	...	1.25	0.29	1.28	...	1.06	1.36
	3.81	...	...	3.25	2.40	2.28	...	1.18	2.89
0.095	1.27	130	2.5	2.45	0.80	2.45	1.63	1.21	2.96
	0.635	...	...	1.50	0.37	2.11	...	1.10	2.32
	3.81	...	...	3.60	2.81	2.90	...	1.25	3.66
0.12	1.27	114	3.1	2.40	0.94	3.01	1.52	1.17	3.53
	0.635	...	...	1.75	0.45	2.71	...	1.13	3.06
0.21	3.81	...	...	4.25	3.77	4.65	...	1.40	6.50
	1.27	98	5.5	2.80	1.23	4.55	1.41	1.28	5.82
	0.635	...	...	1.85	0.55	3.62	...	1.22	4.42
0.30	3.81	...	...	5.00	5.03	7.01	...	1.65	11.5
	1.27	85	7.8	3.00	1.61	6.42	1.32	1.39	8.90
	0.635	...	...	2.10	0.67	5.68	...	1.35	7.82
0.38	3.81	...	...	5.20	5.61	7.71	...	1.84	14.2
	1.27	75	9.9	3.25	1.82	7.52	1.24	1.46	10.7
	0.635	...	...	2.40	0.95	7.16	...	1.35	9.81

[a] Based on experiments performed at Stanford University.

where the term containing μ^2 has been dropped, since its contribution does not exceed 12%.

For comparison with the case of the spherical probe, the ohmic drop due to ion-neutron collisions can now be written in terms of dimensional quantities as

$$\beta k T_e/e = (25\nu_{in}\lambda_D{}^2 I_i)/9LV_{io}{}^{o2} = [(\nu_{in}m_i)/(e^2 N_{e\infty}{}^o Z)](25/36\pi L)I_i$$

The correct value of the plasma density is given, as for the spherical probe, by

$$N_{e\infty}{}^\beta = N_{e\infty}{}^o e^\beta \tag{26}$$

It should be noted that, in contrast with the case of the spherical probe, knowledge of the sheath radius $\rho_o{}^o$ is not required in order to determine the collision correction factor.

IV. Comparison with Experiments

Application of the theory described in the previous two sections involves the following steps: 1) Measurement of the ion saturation current at a normalized biasing potential in excess of $|\eta_p| = 5$. 2) Determination of the electron temperature T_e either from the slope of the volt-ampere characteristic, or from an independent measurement. 3) Evaluation of the hypothetical electron density $N_{e\infty}{}^o$ from the collisionless A.B.R. theory (see Ref. 2). 4) Determination of the ion-neutral collision frequency ν_{in} from ionic mobility considerations 5) Calculation of the correction index α or β from Eqs. (5) or (25), respectively. 6) Determination of the corrected electron density $N_{e\infty}{}^\alpha$ or $N_{e\infty}{}^\beta$ from Eqs. (8) and (26).

This procedure will now be applied to a series of experiments performed by Self and Shih[7] at Stanford University in a gaseous electrical discharge with no mass flow, and to laboratory studies performed by Lederman, Bloom and Widhopf at the Polytechnic Institute of Brooklyn in a slightly ionized hypersonic flow.

A detailed description of the Stanford experiments is presented in Ref. 7. The operating conditions in the positive column of the hot-cathode helium discharge varied in the range $N_{e\infty} = 1.6 \times 10^{10} - 1.4 \times 10^{11}$/cc, $T_e = 75{,}000° - 136{,}000°$K, $T_i \doteq 300°$K, and $p = 0.05 - 0.4$ Torr. Three spherical probes of radii $r_p = 0.089$, 0.047, and 0.026 cm, and three cylindrical probes with $L_p = 1$ cm and radii $\rho_p = 0.038$, 0.013, and 0.0063 cm were placed on the axis of the discharge. The plasma concentrations deduced from the probe characteristics were compared with those deduced from microwave cavity measurements. The data taken with microwave diagnostics and with spherical probes at various values of background pressure and discharge current, as well as the densities obtained from numerical computations are presented in Tables II and III of Ref. 7. The data taken with cylindrical probes, while keeping the discharge current constant at a value $I_D =$ 200 ma and varying the discharge pressure, as well as the results of the calculations according to the present theory are given in Table 1.

It appears from column 6 that ρ_p/λ_D varied in these experiments between 0.3 and 6, and that 15 out of the 18 data points were located in the parametric domain $\rho_p/\lambda_D \leq 3$ where, in the absence of collision, the current drawn by the probe should have been limited by orbital motion. In all these points the measured currents exceeded the predictions of Laframboise's theory by as much as four times.

The plasma concentrations evaluated over a wide range of background pressures and discharge currents are shown in Figs. 2 and 3 for spherical probes and in Fig. 4 for cylindrical probes. In these figures $\bar{N}_{ca}$ denotes the average density, as given by cavity measurements, while $\bar{N}_{ca}(0)$ denotes the

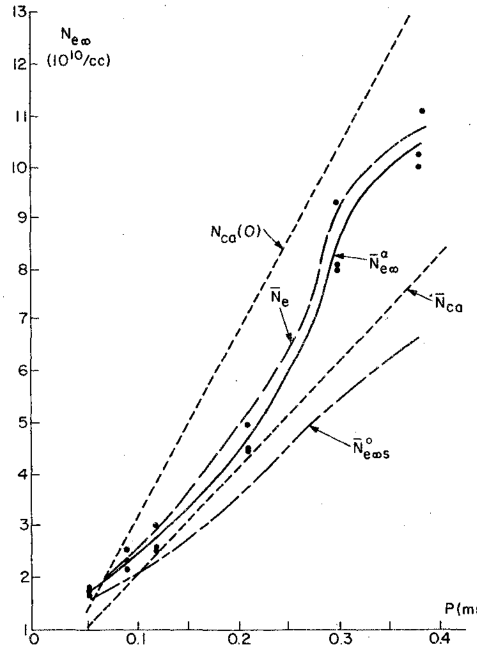

Fig. 2 Plasma concentrations as a function of pressure, (spherical probes, I_D = 200 ma, η_p = −10).

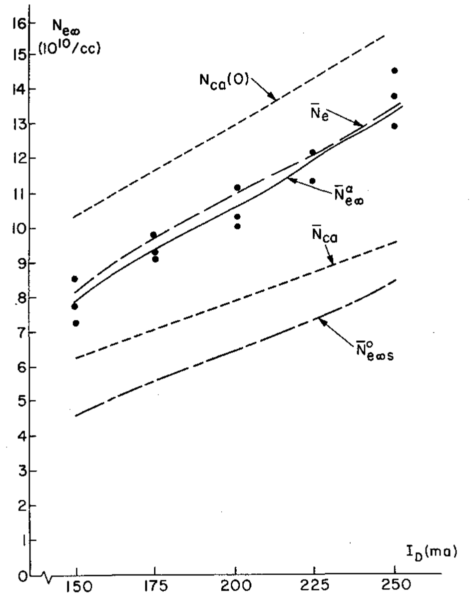

Fig. 3 Plasma concentrations as a function of discharge currents (spherical probes, p = 0.38 mm, η_p = −10).

estimated density along the axis of the column where the probes were located. The evaluation of $N_{ca}(0)$ was based on a standard density profile.[15] $\bar{N}_{e\infty s}{}^o$ and $\bar{N}_{e\infty c}{}^o$ denote the average densities resulting from the application of the collisionless A.B.R. theory to the data obtained from the three spherical and three cylindrical probes, respectively. $\bar{N}_e$ denotes the densities obtained on the basis of the Self and Shih theory[7] for spherical probes. Finally the data points, $\bar{N}_{e\infty}{}^{\alpha}$, and $\bar{N}_{e\infty}{}^{\beta}$, denote the densities and their average values calculated using Eqs. (8) and (26) of this paper for the spherical and cylindrical probes, respectively. These figures indicate that of the 51 data points of different probes, the scattering among 38 of them does not exceed 10% and among the remaining points does not exceed 20%, moreover, the densities evaluated on the basis of the theories presented here for both spherical and cylindrical probes agree with $\bar{N}_e$ within 10%. It must be

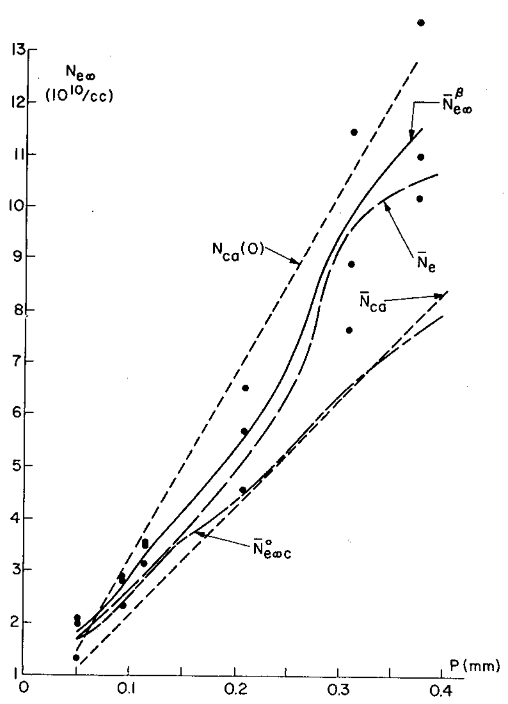

Fig. 4 Plasma concentrations as a function of pressure (cylindrical probes, I_D = 200 ma, η_p = −10).

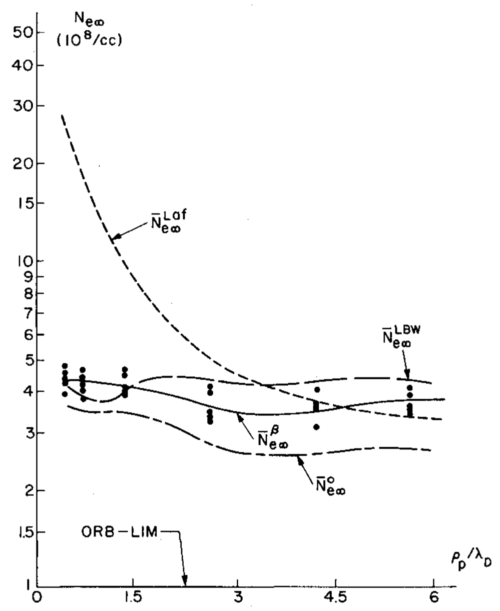

Fig. 5 Plasma concentrations as a function of probe radius (cylindrical probes, η_p = −35).

noted that the current drawn by the probes depresses the plasma density along the axis of the column and this explains why $\bar{N}_e$ is consistently lower than $N_{ca}(0)$, the density on the axis as calculated from cavity measurements. In contrast $\bar{N}_e$ is consistently higher than $\bar{N}_{e\infty}{}^o$, the density given by the A.B.R. theory, especially at the higher background pressures where collisions with neutral become important.

A detailed description of the experiments conducted at PIB is given in Ref. 8. The experiments conducted with cylindrical probes in a shock tunnel were designed to investigate the effects of finite probe length, angle of attack and particularly the validity of Laframboise's orbit theory in a regime where the orbital-motion-limit was presumed to prevail. These tests were recently repeated under more carefully controlled conditions and with more accurate diagnostics, but essentially with the same results. The operating conditions were[16]: Driver gas = air at 38 torr. $T_i \doteq 70°$K, T_e = 3000°K, $N_{e\infty} = 2 - 4 \times 10^8$/cc, $\rho_p = 6.36 \times 10^{-3} - 1.016 \times 10^{-1}$ cm, $L_p/\rho_p = 100$. The experimental results showed that direct application of Laframboise's orbit theory resulted in plasma densities which in the orbital-motion-regime were too high by one order of magnitude. The authors fitted their measured ion currents into an empirical formula[8] (L.B.W.) and obtained a plasma density of about 4×10^8/cc for six different probes in five different runs. This density was confirmed by microwave cavity measurements which yielded a density between 2×10^8 and 4×10^8/cc. The results of our calculations are given in Table 2.

Figure 5 shows the average densities estimated in five runs as a function of the probe radius ρ_p. $\bar{N}_{e\infty}{}^{\mathrm{Laf}}$, $\bar{N}_{e\infty}{}^{\mathrm{LBW}}$, $\bar{N}_{e\infty}{}^o$

Table 2 Cylindrical probe measurements in a hypersonic flow[a]

$\bar{I}\rho_p$ 10^{-2} cm	(η_p = −35) μa	ρ_p/λ_D	$\bar{N}_{e\infty}{}^0$ 10^{10}/cc	e^{β}	$\bar{N}_{e\infty}{}^{\beta}$ 10^{10}/cc
0.635	2.56	0.35	3.78	1.17	4.42
1.27	5.43	0.69	3.55	1.19	4.24
2.54	12.7	1.39	3.57	1.23	4.39
5.08	27.6	2.78	2.83	1.35	3.82
7.62	45.1	4.16	2.69	1.38	3.71
10.16	66.0	5.56	2.70	1.42	3.83

[a] Based on data taken from Ref. 16, T_e = 3,000°K, T_i = 70°K, $\bar{N}e_\infty{}^{LBW}$ = 4.26×10^8/cc, Lp/ρ_p = 100, $\nu_{in} = 8.7 \times 10^4$/sec.

and $\bar{N}_{e\infty}{}^{\beta}$ denote the densities calculated on the basis of Laframboise's, L.B.W.'s, A.B.R.'s and our theory, respectively. It is seen that our theory leads to densities which agree quite well with those deduced on the basis of the L.B.W. formula and confirmed by microwave cavity measurements. The experiments at Stanford and at the Polytechnic cover quite a spectrum of plasma regimes, as can be judged by the following range of parameters $70°\text{K} < T_i < 300°\text{K}$; $3{,}000°\text{K} < T_e < 136{,}000°\text{K}$; $2.1 \times 10^{-3} < T_i/T_e < 2.3 \times 10^{-2}$; $0.05\text{mm} < \lambda_D < 0.2\text{mm}$

$$3 < \lambda_{in}/\lambda_D < 20, \qquad 2 < r_p/\lambda_D < 13, \qquad 25 < L_p/\rho_p < 150$$

$$3 < \lambda_{ii}/\lambda_D < 10, \qquad 0.3 < \rho_p/\lambda_D < 5$$

In addition the ratio of flow velocity to ambipolar sound speed varied from zero to 3.4. Good agreement between the density calculated on the basis of our theory and the density evaluated from microwave interferometry was also obtained for an experiment performed at Cornell University by Dunn and Lordi[17] with $3 < \lambda_{in}/\lambda_D < 5$ and $350 < \lambda_{ii}/\lambda_D < 500$. This means that even when the ion-ion collisions are truly negligible a slight amount of collisions with the neutrals is sufficient to prevent orbital motion and therefore make our theory applicable.

Concluding Remarks

We have described a new method for the evaluation of plasma density by means of ion collection in a regime where collision effects are small but not negligible. This should be interpreted to mean that, to the lowest order in the perturbation scheme, the method applies when 1) the ion-neutral collision frequency is, according to Eqs. (B2) and (B4) of Appendix B

$$\nu_{in} \ll (\tfrac{4}{3})r_o{}^o kT_e\omega_{pi}{}^2/eI_i$$

for a spherical probe

$$\nu_{in} \ll L_p kT_e\omega_{pi}{}^2/eI_i$$

for a cylindrical probe. 2) The reduced mean free path is, according to Eq. (A9) of Appendix A,

$$\lambda_i = (1/\lambda_{ii} + 1/\lambda_{in})^{-1} \leq \rho_p(-e\varphi_p/kT_i)^{1/2}$$

where λ_{ii} is the ion-ion and λ_{in} the ion-neutral mean free path, and φ_p is the probe potential. Under the latter condition there exists enough collisions in the sheath to prevent the orbital motion of the ions. The plasma density can then be found, by multiplying the density evaluated on the basis of the A.B.R. theory by a correction factor e^{α} or e^{β} are given by Eqs. (5) and (25) for the spherical and cylindrical probes, respectively. The corrections thus introduced lead to densities which are in agreement with recently published experimental evidence and, in particular, with two sets of tests performed at Stanford University and at PIB in quite different plasma regimes.

The problem of a spherical probe in a slightly collisional regime has been extensively investigated and numerical solutions are available for comparison.[6] Among these investigators Blue and Ingold,[18] Waymouth[19] and Chou et al.,[20] Kagan and Perel'[21] obtained analytical correction factors for the case of electron collection. Despite the fact that their approaches are quite distinct from one another, they all arrive at the same functional dependence on r_p and λ_{en}, i.e., $1 + \varkappa r_p/\lambda_{en}$ with a slightly different numerical factor $\varkappa$. In the case of small values of α and β our correction factors for ion collection can be similarly expressed as $1 + \alpha$ or $1 + \beta$, with $\alpha \doteq 1.5(T_i/T_e)^{1/2}r_o{}^o/\lambda_{in}$. Thus, we see that in addition to the replacement of the probe radius by the sheath radius, a substitution which does not greatly affect the results, there appears the square root of the ratio between ion and electron temperatures. In the case of interest, when the ions are colder than the electrons, this multiplicative factor reduces the collision correction considerably. As a result we find good agreement with experiment even when the mean free path is a small fraction of the sheath radius and many collisions occur in the sheath.

Appendix A

An estimate of the effectiveness of collisions in impeding orbital motion can be obtained by modifying Langmuir's analysis[22] as follows. We consider the cylindrical probe, where orbital motion is more likely to occur, and we assume that the ions fall freely within a distance from the probe surface equal to a reduced mean free path $\lambda_i = (1/\lambda_{ii} + 1/\lambda_{in})^{-1}$. We assume that the ions enter the cylindrical surface of radius $a = \rho_p + \lambda_i$ with a velocity distribution function in which the portion corresponding to inwardly directed velocities corresponds to the shifted Maxwellian

$$f(u_a,w_a) = N_i\left(\frac{m_i}{2\pi kT_i}\right)\exp\left\{-\frac{m_i[(u_a+u_o)^2+w_a{}^2]}{2kT_i}\right\} \tag{A1}$$

where u_a and w_a are the radial and azimuthal components of the ion velocity vector, and $-u_o$ is the most probable radial velocity. Since most of the potential drop occurs near the probe surface we may assume that the ions fall freely through a potential φ_p. By conservation of energy and angular momentum we then have

$$\begin{aligned} u_a{}^2 + w_a{}^2 &= u_p{}^2 + w_p{}^2 + 2e\varphi_p/m_i \\ aw_a &= \rho_p w_p \end{aligned} \tag{A2}$$

and solving for $u_p{}^2$,

$$u_p{}^2 = u_a{}^2 + w_a{}^2(1 - a^2/\rho_p{}^2) - 2e\varphi_p/m_i$$

The condition that $u_p{}^2$ be positive imposes a maximum value

$$w_a{}^* = (u_a{}^2 - 2e\varphi_p/m_i)^{1/2}(a^2/\rho_p{}^2 - 1)^{-1/2} \tag{A3}$$

for the azimuthal component of velocity of the particle collected by the probe. The total collected current thus becomes

$$I = 2\pi eZN_aL_pa\left(\frac{m_i}{2\pi kT_i}\right)\int_{-\infty}^{0} u_a du_a \int_{-w_a{}^*}^{+w_a{}^*} \exp \times \left\{-\frac{m_i[(u_a+u_o)^2+w_a{}^2]}{2kT_i}\right\} dw_a$$

introducing the notation

$$I' = (\pi kT_i/2\pi eZN_aL_pam_i)I$$

$$u_a + u_o = u_{th}x$$

$$w_a = u_{th}y$$

$$u_{th} = (2kT_i/m_i)^{1/2}$$

we obtain

$$I' = \int_{-\infty}^{u_o/u_{th}} u_{th}{}^3x \int_0^{w_a{}^*/u_{th}} e^{-x^2-y^2}\,dxdy - \int_{-\infty}^{u_o/u_{th}} u_o u_{th}{}^2 \int_0^{w_a{}^*/u_{th}} e^{-x^2-y^2}dxdy \tag{A5}$$

$$= I_1' + I_2'$$

Since $w_a{}^*/u_{th} = (u_a{}^2/u_{th}{}^2 - e\varphi_p/kT_i)^{1/2}(a^2/\rho_p{}^2 - 1)^{-1/2}$ and for a highly negative probe $-e\varphi_p/kT_i \gg T_e/2T_i \doteq u_a{}^2/u_{th}{}^2$, we have

$$w_a{}^*/u_{th} = (-e\varphi_p/kT_i)^{1/2}/(a^2/\rho_p{}^2 - 1)^{1/2} \tag{A6}$$

which is independent of x. Then introducing the error function

$$\text{erf}\,Q = (2/\pi^{1/2})\int_0^Q e^{-y^2}dy$$

we obtain

$$I_1' = (\pi^{1/2}/4)u_{th}{}^3 e^{-u_o{}^2/u_{th}{}^2} \operatorname{erf}(w_a{}^*/u_{th})$$
$$I_2' = -u_{th}{}^2 u_o \pi/4[1 + \operatorname{erf}(u_o/u_{th})] \operatorname{erf}(w_a{}^*/u_{th}) \tag{A7}$$

We observe that $|I_2'| \gg |I_1'|$ and since $u_o/u_{th} \gg 1$, $\operatorname{erf}(u_o/u_{th}) = 1$. Therefore, the collected current is approximately

$$I \doteq -2\pi e Z N_a L_p a u_o \operatorname{erf}(-e\varphi_p/kT_i)/(a^2/\rho_p{}^2 - 1) \tag{A8}$$

Since the error function assumes a value of 0.84 when the argument is 1 we conclude that the effect of orbital motion can be neglected when

$$[(a^2/\rho_p{}^2) - 1]^{1/2} \leq [-(e\varphi_p/kT_i)]^{1/2}$$

or

$$(\lambda_i/\rho_p) \leq [-(e\varphi_p/kT_i)]^{1/2} \tag{A9}$$

Equation (A9) establishes a criterion for the applicability of our theory and was satisfied in all the experiments considered.

It is worthwhile mentioning that the assumptions leading to Eq. (A8) are valid only for the case under consideration, in which the mean free path is in the order of the sheath radius, and, therefore, the most probable radial velocity u_o is in the order of $(kT_e/m_i)^{1/2}$. In contrast, the values of u_o appropriate to the collisionless and collision-dominated limiting cases are $(kT_i/m_i)^{1/2}$ and $(-e\varphi_p/m_i)^{1/2}$, respectively. For this reason one should not expect the asymptotic forms of Eq. (A8) to yield the orbital-motion and continuum limit currents.

Appendix B

In our analysis we develop the perturbation scheme only to the lowest order. This sets an upper bound for the collision frequency ν_{in} that can be evaluated as follows. In substituting the collisionless v^o for v in the integration which leads to the determination of the collision correction factors, Eqs. (4) and (19), we neglect the first leading term containing γ. In the case of the spherical probe this implies that

$$(\tfrac{1}{4})a_3 x_o{}^{o4}/(\tfrac{1}{3})a_2 x_o{}^{o3} = (3a_3/4a_2)x_o{}^o \ll 1 \tag{B1}$$

or

$$(\tfrac{3}{4})(2^{1/2}J)^{1/2}\gamma x_o{}^o = (\tfrac{3}{4})\alpha \ll 1$$

and in dimensional form

$$\nu_{in} \ll (\tfrac{4}{3})(r_o{}^o kT_e\omega_{pi}{}^2/eI_i) \tag{B2}$$

Similarly for the cylindrical probe we obtain

$$2^{1/2}\gamma J\lambda_D/L_p = (\tfrac{9}{25})\beta \ll 1 \tag{B3}$$

and in dimensional form

$$\nu_{in} \ll L_p kT_e\omega_{pi}{}^2/eI_i \tag{B4}$$

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